LESSON 21.3

Using Special Factors to Solve Equations

Common Core Math Standards

The student is expected to:

CA CC **A-SSE.3a**

Factor a quadratic expression to reveal the zeros of the function it defines. Also A-SSE.2, A-REI.4b, F-LE.6

Mathematical Practices

CA CC **MP.1 Problem Solving**

Language Objective

Explain to a partner what a perfect-square trinomial is and how you can recognize one.

ENGAGE

Essential Question: How can you use special products to aid in solving quadratic equations by factoring?

By recognizing that a polynomial is a perfect-square trinomial or a difference of squares, you can use the appropriate special product rule to factor the polynomial, and then use the zero product property to solve the equation.

PREVIEW: LESSON PERFORMANCE TASK

View the Engage section online. Discuss how the shape of the base of a fountain could affect the possible patterns made by the falling water. Then preview the Lesson Performance Task.

Name ____________ Class ________ Date ______

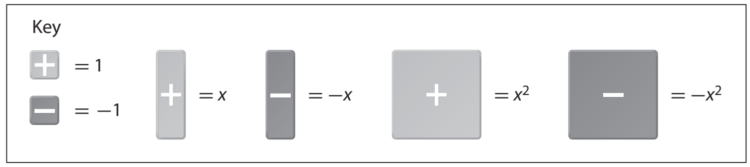
21.3 Using Special Factors to Solve Equations

Resource Locker

Essential Question: How can you use special products to aid in solving quadratic equations by factoring?

Explore Exploring Factors of Perfect Square Trinomials

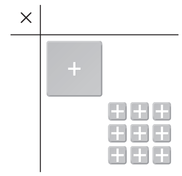
When you use algebra tiles to factor a polynomial, you must arrange the unit tiles on the grid in a rectangle. Sometimes, you can arrange the unit tiles to form a square. Trinomials of this type are called perfect-square trinomials.



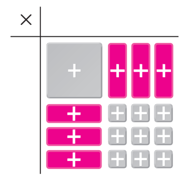
Ⓐ Use algebra tiles to factor $x^2 + 6x + 9$.

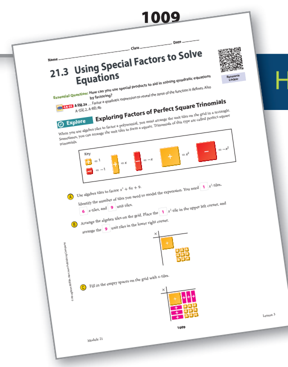
Identify the number of tiles you need to model the expression. You need **1** x^2-tiles, **6** x-tiles, and **9** unit tiles.

Ⓑ Arrange the algebra tiles on the grid. Place the **1** x^2-tile in the upper left corner, and arrange the **9** unit tiles in the lower right corner.



Ⓒ Fill in the empty spaces on the grid with x-tiles.





HARDCOVER PAGES 791–800

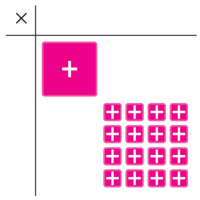
Turn to these pages to find this lesson in the hardcover student edition.

Ⓓ All **6** x-tiles were used, so all the tiles are accounted for and fit in the square with sides of length **$x+3$**. Read the length and width of the square to get the factors of the trinomial $x^2+6x+9=(x+\mathbf{3})(x+\mathbf{3})$.

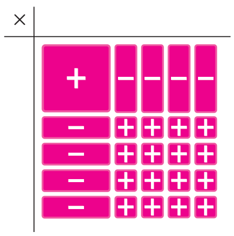
Ⓔ Now, use algebra tiles to factor $x^2-8x+16$.

You need **1** x^2-tiles, **8** $-x$ -tiles, and **16** unit tiles to model the expression.

Ⓕ Arrange the algebra tiles on the grid. Place the **1** x^2-tile in the upper left corner, and arrange the **16** unit tiles in the lower right corner.



Ⓖ Fill in the empty spaces on the grid with $-x$-tiles.



Ⓗ All **8** $-x$ -tiles were used, so all the tiles are accounted for and fit in a square with sides of length **$x-4$**. Read the length and width of the square to get the factors of the trinomial $x^2-8x+16=(\mathbf{x-4})(\mathbf{x-4})$.

Reflect

1. **What If?** Suppose that the middle term in x^2+6x+9 was changed from $6x$ to $10x$. How would this affect the way you factor the polynomial?
The arrangement of unit tiles would have to be in a rectangle, not a square. The factored form $x^2+10x+9$ is $(x+1)(x+9)$.

EXPLORE

Exploring Factors of Perfect-Square Trinomials

INTEGRATE TECHNOLOGY

Students have the option of completing the algebra tiles activity either in the book or online.

INTEGRATE MATHEMATICAL PRACTICES

Focus on Modeling

MP.4 Remind students that modeling the factors of trinomials with algebra tiles means the tiles must be arranged in a rectangle. When the factors of a perfect-square trinomial are modeled with algebra tiles, the tiles will be arranged in a square.

PROFESSIONAL DEVELOPMENT

Learning Progressions

In this lesson, students expand their understanding of factoring trinomials of the form ax^2+bx+c by learning to recognize and factor perfect-square trinomials, the difference of two squares, and polynomials that consist of one of these special products multiplied by a monomial factor. They also solve equations and real-world problems that involve these polynomials. As they work with polynomial expressions that follow certain patterns, students learn to recognize the patterns and apply them when appropriate. The ability to factor special products efficiently will be valuable in algebra as well as in future courses that require repeated reasoning with polynomials.

EXPLAIN 1

Factoring $a^2x^2 + 2abx + b^2$ and $a^2x^2 - 2abx + b^2$

QUESTIONING STRATEGIES

How can you tell whether a trinomial of the form $ax^2 + bx + c$ is a perfect-square trinomial? **If it is a perfect-square trinomial, both *a* and *c* will be perfect squares, and *b* will be equal to twice the product of the square roots of *a* and *c*, or twice the opposite of that product.**

AVOID COMMON ERRORS

When asked to factor a trinomial that has a common monomial factor, such as $18x^2 - 60x + 50$, students may see that the first and last terms are not perfect squares and therefore assume that they cannot use the rule for factoring perfect squares. Remind students to always begin by factoring out any common factors, and then examine the trinomial that remains to decide whether they can use a special product rule.

2. If the positive unit squares are arranged in a square of unit tiles when factoring with algebra tiles, what will be true about the binomial factors? (The coefficient of the x^2 term is 1 as in the previous problems.)
Both factors will be the same, as in $(x+3)(x+3)$.

Explain 1 Factoring $a^2x^2 + 2abx + b^2$ and $a^2x^2 - 2abx + b^2$

Recall that a perfect-square trinomial can be represented algebraically in either the form $a^2 + 2ab + b^2$ or the form $a^2 - 2ab + b^2$.

Perfect-Square Trinomials

Perfect-Square Trinomial	Examples
$a^2 + 2ab + b^2 = (a+b)(a+b) = (a+b)^2$	$x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$
	$c^2x^2 + 2cdx + d^2 = (cx)^2 + 2cdx + d^2 = (cx+d)(cx+d) = (cx+d)^2$
$a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$	$x^2 - 10x + 25 = (x-5)(x-5) = (x-5)^2$
	$c^2x^2 - 2cdx + d^2 = (cx)^2 - 2cdx + d^2 = (cx-d)(cx-d) = (cx-d)^2$

Example 1 **Factor perfect-square trinomials.**

(A) $4x^3 - 24x^2 + 36x$

$4x^3 - 24x^2 + 36x = 4x(x^2 - 6x + 9)$ — Factor out the common monomial factor $4x$.

$= 4x\,[x^2 - 2(1 \cdot 3)x + 3^2]$ — Rewrite the perfect square trinomial in the form $a^2x^2 - 2abx + b^2$.

$= 4x(x-3)(x-3)$ — Rewrite the perfect square trinomial in the form $(ax-b)(ax-b)$ to obtain factors.

The factored form of $4x^3 - 24x^2 + 36x$ is $4x(x-3)(x-3)$, or $4x(x-3)^2$.

(B) $x^2 + 16x + 64$

$x^2 + 16x + 64 = x^2 + 2(\boxed{1} \cdot \boxed{8})x + \boxed{8}^2$ — Rewrite in the form $a^2x^2 + 2abx + b^2$.

$= (x + \boxed{8})(x + \boxed{8})$ — Rewrite in the form $(ax+b)(ax+b)$.

The factored form of $x^2 + 16x + 64$ is $(x + \boxed{8})(x + \boxed{8})$, or $(x + \boxed{8})^2$.

COLLABORATIVE LEARNING

Peer-to-Peer Activity

Have students work in pairs. Ask each student to write five polynomials, each of which is a perfect-square trinomial or the difference of two squares. Students then trade polynomials and factor the ones they receive. Finally, ask students to explain to each other what steps they used to factor each polynomial.

Your Turn

Factor perfect-square trinomials.

3. $2y^3 + 12y^2 + 18y$

$$\begin{aligned} 2y^3 + 12y^2 + 18y &= 2y(y^2 + 6y + 9) \\ &= 2y[y^2 + 2(1 \cdot 3)y + 3^2] \\ &= 2y(y + 3)(y + 3) \\ &= 2y(y + 3)^2 \end{aligned}$$

4. $100z^2 - 20z + 1$

$$\begin{aligned} 100z^2 - 20z + 1 &= 10^2z^2 - 2(10 \cdot 1)z + 1^2 \\ &= (10z - 1)(10z - 1) \\ &= (10z - 1)^2 \end{aligned}$$

Explain 2 Factoring $a^2x^2 - b^2 = 0$

Recall that a difference of squares can be written algebraically as $a^2 - b^2$ and factored as $(a + b)(a - b)$.

Difference of Squares

Difference of Two Squares	
Perfect-Square Trinomial	**Examples**
$a^2 - b^2 = (a + b)(a - b)$	$x^2 - 9 = (x + 3)(x - 3)$ $4x^2 - 9 = (2x + 3)(2x - 3)$ $9x^2 - 1 = (3x + 1)(3x - 1)$ $c^2x^2 - d^2 = (cx)^2 - d^2$ $= (cx + d)(cx - d)$

Example 2 Factor each difference of squares.

(A) $x^2 - 49$

$x^2 - 49 = x^2 - 7^2$ Rewrite in the form $a^2x^2 - b^2$.

$= (x + 7)(x - 7)$ Rewrite in the form $(ax + b)(ax - b)$.

The factored form of $x^2 - 49$ is $(x + 7)(x - 7)$.

(B) $49q^2 - 4p^2$

$49q^2 - 4p^2 = \boxed{7}^2\boxed{q}^2 - \left(\boxed{2p}\right)^2$ Rewrite in the form $a^2x^2 - b^2$.

$= \left(\boxed{7q + 2p}\right)\left(\boxed{7q - 2p}\right)$ Rewrite in the form $(ax + b)(ax - b)$.

The factored form of $49q^2 - 4p^2$ is $\left(\boxed{7q + 2p}\right)\left(\boxed{7q - 2p}\right)$.

EXPLAIN 2

Factoring $a^2x^2 - b^2 = 0$

AVOID COMMON ERRORS

Students might think that y^6 is not a perfect square because the number 6 is not a perfect square. Remind them of the Product of Powers Property, which states that $a^m \cdot a^n = a^{m+n}$. Thus, y^6 is a perfect square because it can be written as $y^3 \cdot y^3$. The Power of a Power Property can also be used to show this as $y^6 = (y^3)^2$.

QUESTIONING STRATEGIES

Can you factor $x^2 + 25$ as a difference of two squares? Explain. **No; it is a sum of two squares, not a difference. The operation sign for the difference of two squares must be −.**

What are the values of a and b in the difference of squares $9 - x^4$? **$a = 3$ and $b = x^2$**

INTEGRATE MATHEMATICAL PRACTICES

Focus on Reasoning

MP.2 Discuss with students what happens if you forget to factor out a common factor of the terms in a difference of two squares. For example, to factor $4x^2 - 16y^4$, you could first factor out a common factor of 4 to get $4(x^2 - 4y^4)$, then factor the difference of squares to get $4(x + 2y^2)(x - 2y^2)$. If you do not notice the common factor, you can still factor the expression as a difference of squares, to get $(2x + 4y^2)(2x - 4y^2)$. Help students to see that this would be an equivalent expression, but it would not be a complete factorization. You would still need to factor a 2 out of each binomial to get the same final result as before.

DIFFERENTIATE INSTRUCTION

Number Sense

Show students a quick way to rule out the possibility that a trinomial is a perfect-square trinomial. Remind them that if $ax^2 + bx + c$ is a perfect-square trinomial, then b must be an even number. This is because, in a perfect-square trinomial, $b = 2 \cdot \sqrt{a} \cdot \sqrt{b}$. If b is odd, then the trinomial is not a perfect square.

EXPLAIN 3

Solving Equations with Special Factors

INTEGRATE MATHEMATICAL PRACTICES

Focus on Patterns

MP.8 When students factor a perfect-square trinomial, they should first look at the sign of the x-term, as this will tell them which pattern to use. If it is $+$, they should use $(a+b)^2$; if it is $-$, they should use $(a-b)^2$.

QUESTIONING STRATEGIES

? How are the equations $25x^2 + 20x + 4 = 0$ and $75x^2 + 60x + 12 = 0$ related? How do their solutions compare? Explain. **The second equation is equal to the first one multiplied by 3. Both equations have the same solution, $x = -\frac{2}{5}$. The common factor of 3 in the second equation has no effect on the solution, because when you divide both sides of the equation by 3, it is identical to the first equation.**

Reflect

5. **Discussion** James was factoring a difference of squares but did not finish his work. What steps is he missing?

$$16x^4 - 1 = (4x^2)^2 - 1$$
$$= (4x^2+1)(4x^2-1)$$

James factored the first difference of squares, $16x^4 - 1$, but forgot to factor the second one that came up, $4x^2 - 1$. This is his completed work:

$$16x^4 - 1 = (4x^2)^2 - 1$$
$$= (4x^2+1)(4x^2-1)$$
$$= (4x^2+1)\left[(2x)^2 - 1\right]$$
$$= (4x^2+1)(2x+1)(2x-1)$$

Your Turn

Factor each difference of squares.

6. $x^2 - 144$

$$x^2 - 144 = x^2 - 12^2$$
$$= (x+12)(x-12)$$

7. $81y^4 - 9y^2$

$$81y^4 - 9y^2 = 9y^2(9y^2 - 1)$$
$$= 9y^2(3^2y^2 - 1)$$
$$= 9y^2(3y+1)(3y-1)$$

Explain 3 Solving Equations with Special Factors

Equations with special factors can be solved using the Zero Product Property. Remember, the Zero Product Property states that if the product of two factors is zero, then at least one of the factors must be zero. For example, if $(x+1)(x+9) = 0$ then $x + 1 = 0$ or $x + 9 = 0$. Consequently, the solutions for the equation are $x = -1$ or $x = -9$.

Example 3 **Solve the following equations with special factors.**

Ⓐ $4x^2 + 12x + 9 = 0$

$4x^2 + 12x + 9 = 0$	
$2^2x^2 + 2(2 \cdot 3)x + 3^2 = 0$	Rewrite in the form $a^2x^2 + 2abx + b^2$.
$(2x+3)(2x+3) = 0$	Rewrite in the form $(ax+b)(ax+b)$.
$2x + 3 = 0$	Set factors equal to 0 using Zero Product Property.
$x = -\frac{3}{2}$	Solve equation.

LANGUAGE SUPPORT EL

Modeling

Students may know and use the word *perfect* to mean flawless, but in mathematics the term *perfect square* has a specific meaning. A number that is a perfect square is *the square of a whole number,* and a *perfect-square trinomial* is *the square of a binomial.*

Demonstrate how to model the factors of trinomials with algebra tiles. Point out that if a trinomial can be factored, the tiles form a rectangle. For a perfect-square trinomial, the tiles will be arranged in a square. Be sure to use and repeat the terminology as you work together so that students hear the words in context, which helps clarify their meaning.

(B) $25x^2 - 1 = 0$

$25x^2 - 1 = 0$

$\boxed{5}^2 x^2 - \boxed{1}^2 = 0$ Rewrite in the form $a^2x^2 - b^2$.

$\left(\boxed{5x+1}\right)\left(\boxed{5x-1}\right) = 0$ Rewrite in the form $(ax+b)(ax-b)$.

$\boxed{5x-1} = 0$ or $\boxed{5x+1} = 0$ Set factors equal to 0 using Zero Product Property.

$x = \boxed{\frac{1}{5}}$ or $x = \boxed{-\frac{1}{5}}$ Solve equation.

Your Turn

Solve the following equations with special factors.

8. $25x^2 - 10x + 1 = 0$

$$25x^2 - 10x + 1 = 0$$
$$5^2x^2 - 2(5 \cdot 1)x + 1^2 = 0$$
$$(5x-1)(5x-1) = 0$$
$$5x - 1 = 0$$
$$5x = 1$$
$$x = \frac{1}{5}$$

9. $8x^4 - 2x^2 = 0$

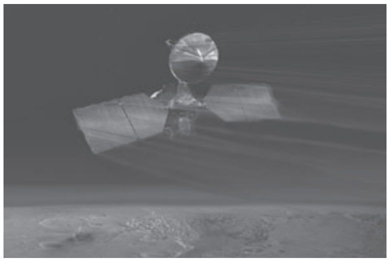
$$8x^4 - 2x^2 = 0$$
$$2x^2(4x^2 - 1) = 0$$
$$2x^2(2^2x^2 - 1) = 0$$
$$2x^2(2x+1)(2x-1) = 0$$
$$x^2 = 0 \quad \text{or} \quad 2x + 1 = 0 \quad \text{or} \quad 2x - 1 = 0$$
$$x = 0 \qquad 2x = -1 \qquad 2x = 1$$
$$x = 0 \quad \text{or} \quad x = -\frac{1}{2} \quad \text{or} \quad x = \frac{1}{2}$$

Explain 4 Solving Equation Models with Special Factors

For each real-world scenario, solve the model which involves an equation with special factors.

Example 4 **Write the given information and manipulate into a familiar form. Solve the equation to answer a question about the situation.**

As a satellite falls from outer space onto Mars, its distance in miles from the planet is given by the formula $d = -9t^2 + 776$, where t is the number of hours it has fallen. Find when the satellite will be 200 miles away from Mars.



Analyze Information

Identify the important information

- The satellite's distance in miles is given by the formula $\underline{d = -9t^2 + 776}$.
- The satellite distance at some time t is $d = \boxed{200}$.

EXPLAIN 4

Solving Equation Models with Special Factors

AVOID COMMON ERRORS

Remind students that the Zero Product Property can be applied only to quadratic equations in standard form, because it applies only to equations for which one side is zero. Students often try to apply it before setting one side equal to zero.

QUESTIONING STRATEGIES

If the equation that models a real-world problem has the form $x^2 - b^2 = 0$ for some constant b, why might you need to discard one solution for x? **Factoring $x^2 - b^2$ gives $(x + b)(x - b)$, so the solutions to the equation are $x = b$ and $x = -b$. One of these values is positive and one is negative. If x represents a quantity such as time or distance, a negative value may not make sense in the context of the problem.**

Formulate a Plan

Substituting the value of the constant $d = \boxed{200}$ into the equation $\underline{d = -9t^2 + 776}$ you get the equation $\underline{200 = -9t^2 + 776}$. Simplify the new equation into a familiar form and solve it.

Solve

Rewrite the equation to be equal to 0.

$\boxed{200} = -9t^2 + 776$ Subtract 200 from both sides.

$0 = -9t^2 + \boxed{576}$ Divide both sides by -1.

$0 = 9t^2 - \boxed{576}$ Factor out 9.

$0 = \boxed{9}\left(t^2 - \boxed{64}\right)$

The equation contains a $\underline{\text{difference of squares}}$ that you can factor.

$0 = 9\left(\boxed{t + 8}\right)\left(\boxed{t - 8}\right)$

Use the $\underline{\text{Zero Product}}$ Property to solve.

$0 = 9\left(t + \boxed{8}\right)\left(t - \boxed{8}\right)$

$\boxed{t + 8} = 0$ or $t - 8 = \boxed{0}$

$t = \boxed{\pm 8}$

The answer is $t = \boxed{8}$ because time must be $\underline{\text{positive}}$. So, the satellite has fallen for $\boxed{8}$ hours.

Justify and Evaluate

$t = \boxed{8}$ makes sense because time must be $\underline{\text{positive}}$. Check by substituting this value of t into the original equation.

$-9 \cdot \boxed{8}^2 + 776 = -9 \cdot \boxed{64} + 776$

$= 776 - \boxed{576}$

$= \boxed{200}$

This is what is expected from the given information.

Your Turn

Write the given information and manipulate it into a familiar form. Solve the equation to answer a question about the situation.

10. A volleyball player sets the ball in the air, and the height of the ball after t seconds is given in feet by $h = -16t^2 + 12t + 6$. A teammate wants to wait until the ball is 8 feet in the air before she spikes it. When should the teammate spike the ball? How many reasonable solutions are there to this problem? Explain.

$$-16t^2 + 12t + 6 = 8$$
$$-16t^2 + 12t - 2 = 0$$
$$8t^2 - 6t + 1 = 0$$
$$(4t - 1)(2t - 1) = 0$$
$$4t - 1 = 0 \quad \text{or} \quad 2t - 1 = 0$$
$$t = \frac{1}{4} \qquad t = \frac{1}{2}$$

At 0.25 second or 0.5 second the ball will be 8 ft high. There are two solutions because both occur after $t = 0$, when the ball is set.

11. The height of a model rocket is given (in centimeters) by the formula $h = -490t^2$, where t is measured in seconds and $h = 0$ refers to its original height at the top of a mountain. It begins to fly down from the mountain-top at time $t = 0$. When has the rocket descended 490 centimeters?

$$-490 = -490t^2$$
$$490t^2 - 490 = 0$$
$$490(t^2 - 1) = 0$$
$$490(t + 1)(t - 1) = 0$$
$$t = \pm 1$$

After one second, the rocket will have descended a distance of 490 centimeters. The negative time cannot be used in this context.

Elaborate

12. Are the perfect square trinomials $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ very different? How can you get one from the other?

Start with the first form $a^2 + 2ab + b^2$. Let $b = -b_{new}$ and simplify.

$$a^2 + 2ab + b^2 = a^2 + 2a(-b_{new}) + (-b_{new})^2$$
$$= a^2 - 2ab_{new} + \left[(-1)(b_{new})\right]^2$$
$$= a^2 - 2ab_{new} + \cancel{(-1)^2}(b_{new})^2$$
$$= a^2 - 2ab_{new} + {b_{new}}^2$$

$a^2 - 2ab_{new} + {b_{new}}^2$ is the other form. They are not very different.

ELABORATE

QUESTIONING STRATEGIES

? What types of quadratic equations have only one real solution? **Quadratic equations that can be factored as perfect squares have only one real solution. For example, $4x^2 - 16x + 16 = 0$, which can be factored as $4(x - 2)^2 = 0$, has one real solution, $x = 2$.**

SUMMARIZE THE LESSON

What are two characteristics of a perfect-square trinomial of the form $ax^2 + bx + c$? **a and c are perfect squares, and $b = 2\sqrt{ac}$.**

What are two characteristics of a difference of squares? **Both terms are perfect squares, and one term is subtracted from the other.**

13. How would you go about factoring $a^2 - 2ab + b^2 - 1$?

There is a perfect-square trinomial inside the expression that you can factor. There is also a difference of squares.

$$a^2 - 2ab + b^2 - 1 = (a - b)^2 - 1^2$$
$$= (a - b + 1)(a - b - 1)$$

14. Setting a perfect-square trinomial equal to zero, $a^2x^2 + 2abx + b^2 = 0$, produces how many solutions? How many solutions are produced setting a difference of squares equal to zero, $a^2x^2 - b^2 = 0$?

$$a^2x^2 + 2abx + b^2 = 0$$
$$(ax + b)^2 = 0$$
$$ax = -b$$
$$x = -\frac{b}{a}$$

$a^2x^2 + 2abx + b^2 = 0$ produces one solution for x.

$$a^2x^2 - b^2 = 0$$
$$(ax + b)(ax - b) = 0$$
$$ax = \pm b$$
$$x = \pm\frac{b}{a}$$

$a^2x^2 - b^2 = 0$ produces two solutions for x.

15. Physical problems involving projectile motion can be modeled using the general equation $h = -16t^2 + v_0t$. Here, h refers to the relative height of the projectile from its initial position, v_0 is its initial vertical velocity, and t is time elapsed from launch. If you are measuring the height of the projectile as it descends from a high place, and it was launched with $v_0 = 0$ (which means it was thrown horizontally or dropped), how would you use special products to find the time at which it reaches a given height? (Assume that the height the projectile has descended is a square number in this question, although this is not a requirement in real life).

Set $v_0 = 0$ and $h = -s^2$ for some value of s. h is negative because the projectile is descending from its original position. Substitute these values into the equation and simplify.

$$h = -16t^2 + v_0t$$
$$h = -16t^2$$
$$-s^2 = -16t^2$$
$$16t^2 - s^2 = 0$$

At this point you can use the difference of squares to solve.

$$(4t + s)(4t - s) = 0$$
$$4t = \pm s$$
$$t = \pm\frac{s}{4} = \pm\frac{\sqrt{-h}}{4}$$

Only the positive answer applies to the model, so the solution is $t = \frac{\sqrt{-h}}{4}$. The difference of squares was used to find this.

16. Essential Question Check-In How can you use special products to solve quadratic equations?

Once you recognize that a polynomial is a perfect-square trinomial or a difference of squares, you can factor the polynomial. Then, use the Zero Product Property to solve the equation.